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LETTER TO THE EDITOR

Conformal invariance and universality in finite-size scaling

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Abstract. The universal relation between critical exponents and the amplitude of the correlation length divergence as a function of finite size at the critical point of two-dimensional systems is shown to be a consequence of conformal invariance. Both periodic and free boundary conditions are considered.

In studies of the finite-size scaling behaviour of two-dimensional systems on infinitely long strips of finite width, a remarkable universality has been observed. The theory of finite-size scaling (Barber 1983) predicts that the inverse correlation length $\varkappa_n$ (measured in lattice spacings), for a strip of width n lattice spacings, should behave, at the critical point of the infinite system, as

$$\varkappa_n \sim A/n \tag{1}$$

where the amplitude A is universal in the usual sense (Privman and Fisher 1984). However, exact and numerical calculations (Luck 1982, Derrida and de Seze 1982, Nightingale and Blöte 1983, Privman and Fisher 1984) also suggest that the ratio A/x, where x is the scaling dimension of the operator concerned, is equal to 2π for several different operators in a wide variety of isotropic two-dimensional models.

In this letter, we point out that this result is a simple consequence of conformal covariance of the correlation functions, which is believed to hold in the continuum limit at the critical point (Polyakov 1970, 1974). In two dimensions, conformal invariance implies that the correlation functions of a local scalar operator $\varphi(z)$ satisfy

$$\langle\varphi(z_1)\varphi(z_2)\rangle = |w'(z_1)|^x|w'(z_2)|^x\langle\varphi(w(z_1))\varphi(w(z_2))\rangle \tag{2}$$

where $z\to w(z)$ is an arbitrary conformal transformation, and x is the scaling dimension of φ. Now consider the particular transformation

$$w = \ln z \tag{3}$$

which maps the whole z-plane into the strip $|\mathrm{Im}\, w| \leq \pi$, with periodic boundary conditions. Using the result that in the infinite plane

$$\langle\varphi(z_1)\varphi(z_2)\rangle \sim |z_1 - z_2|^{-2x} \tag{4}$$

and writing $z_j = \exp(y_j + \mathrm{i}\theta_j)$, $(|\theta_j| \leq \pi)$, it follows from (2) that

$$\langle\varphi(y_1+\mathrm{i}\theta_1)\varphi(y_2+\mathrm{i}\theta_2)\rangle_S \sim \exp\{-x\ln[\exp(y_1-y_2)+\exp(y_2-y_1)-2\cos(\theta_1-\theta_2)]\} \tag{5}$$

where the correlation function on the left-hand side is evaluated in the strip geometry.

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The inverse correlation length $\varkappa$ is defined by

$$\int_{-\pi}^{\pi} \mathrm{d}\theta_1 \langle \varphi(y_1+\mathrm{i}\theta_1)\varphi(y_2+\mathrm{i}\theta_2)\rangle_{\mathrm{S}} \sim \exp(-\varkappa|y_1-y_2|) \tag{6}$$

as $|y_1-y_2|\to\infty$. Comparing with (5) we see that $\varkappa = x$.

However, this is measured in physical units in which the width of the strip is 2π. If we now introduce a lattice, the lattice spacing will be $2\pi/n$. Hence $\varkappa_n$ as measured in inverse lattice spacings equals $2\pi x/n$, which is the desired result.

The same methods may be applied to the case of free boundary conditions. The same transformation (3) maps the half plane $\mathrm{Re}\, z \geqslant 0$ onto the strip $|\mathrm{Im}\, w| \leqslant \frac{1}{2}\pi$, with the same boundary conditions. We then find

$$\langle \varphi(y_1+\mathrm{i}\theta_1)\varphi(y_2+\mathrm{i}\theta_2)\rangle_{\mathrm{S}} \sim \exp[x(y_1+y_2)]\langle \varphi[\exp(y_1+\mathrm{i}\theta_1)]\varphi[\exp(y_2+\mathrm{i}\theta_2)]\rangle_{\mathrm{S}'} \tag{7}$$

where S′ refers to the surface geometry. The theory of surface critical phenomena (Binder 1983) now implies that the right-hand side has the scaling form† $F(\exp(y_1-y_2), \theta_1, \theta_2)$, where, for $\mathrm{e}^{y_1} \gg \mathrm{e}^{y_2}$,

$$F(\exp(y_1-y_2), \theta_1, \theta_2) \sim a(\theta_1, \theta_2)\exp[-x_{\mathrm{s}}(y_1-y_2)] \tag{8}$$

where x_{s} is the scaling dimension of the corresponding surface operator (equal to $\frac{1}{2}\eta_{\|}$ for the order parameter). If we now define the inverse correlation length in the strip by integrating (7) over θ_1 and θ_2 we find $\varkappa = x_{\mathrm{s}}$. Since the width in physical units is now π, the inverse correlation length measured in inverse lattice spacings is

$$\varkappa_n \sim \pi x_{\mathrm{s}}/n \tag{9}$$

or, equivalently, $\varkappa_n \sim \pi\eta_{\|}/2n$ for the order parameter correlation length.

The above calculations may be easily verified for the Gaussian model, where conformal invariance is explicit. In that case, $\eta_{\|} = 2\eta$, so the amplitude A is the same for the two different boundary conditions. For the Ising model, however, $\eta_{\|} = 1$, so we predict $\varkappa_n \sim \pi/2n$ for free boundaries.

Finally, we mention that recently Friedan *et al* (1983) have used conformal invariance and unitarity to constrain severely possible exponents in two dimensions. The requirement of unitarity appears to exclude several interesting models (percolation, continuous q-state Potts, continuous N-component cubic models), for which the universality of A/x has been numerically verified (Derrida and de Seze 1982, Nightingale and Blöte 1983). This suggests that such models are conformally invariant but not unitary.

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References

Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8 ed C Domb and J Lebowitz (London: Academic)

Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8 ed C Domb and J Lebowitz (London: Academic)

† Conformal invariance may also be used to constrain the form of F.

Derrida B and de Seze J 1982 *J. Physique* **43** 475
Friedan D, Qiu Z and Shenker S 1983 *Talk presented at Workshop on Vertex Operators, Berkeley* to be published
Luck J M 1982 *J. Phys. A: Math. Gen.* **15** L169
Nightingale M P and Blöte H W J 1983 *J. Phys. A: Math. Gen.* **16** L657
Polyakov A M 1970 *Pisma ZhETP* **12** 538 (1970 *JETP Lett.* **12** 381)
—— 1974 *ZhETP* **66** 23 (1974 *JETP Lett.* **39** 10)
Privman V and Fisher M E 1984 *Phys. Rev.* B in print